ISRAEL JOURNAL OF MATHEMATICS, Vol. 29, No. 1, 1978

UNIVERSAL RADIUS OF INJECTIVITY FOR LOCALLY QUASICONFORMAL MAPPINGS

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ABSTRACT

If $n > 2$ and if f is a locally quasiconformal mapping from the ball $B^n = \{x \in R^n : |x| < 1\}$ into $R^n \cup \{\infty\}$ then f is injective in $B^n(r) = \{x \in R^n : |x| < r\}$ where $r > 0$ depends only on n and the maximal dilatation of f.

1. Introduction

In this note we consider local homeomorphisms f of bounded dilatation from the unit ball $B^n = \{x \in R^n : |x| < 1\}$ into $\bar{R}^n = R^n \cup \{\infty\}$. In [6, theor. 2.3] it was shown that if $n \geqq 3$ and $fB^n \subset R^n$, then f is one to one in $B^n(r) = \{x \in R^n : |x| < r\}$ where r depends only on n and the maximal dilatation of f. The complex functions $z \mapsto (z-1)^m$, $m = 1, 2, \cdots$, show that the result is false for $n = 2$. It has been an open problem whether the condition $fB^n \subset R^n$ is dispensable. Here we give an affirmative answer to this question and prove

1.1. THEOREM. *Suppose that $f: B^n \to \bar{R}^n$ is a quasimeromorphic local homeomorphism. If $n > 3$ then f is injective in $B^n(r)$ where r depends only on n and the maximal dilatation $K(f)$ of f.*

The weaker version [6, theor. 2.3] of the above theorem has been used by several authors to study various properties of quasiregular mappings, see [2, theor. 3], [6, corol. 2.7, 2.8, 2.10 and theor. 2.9], [7, theor. 6.12], and [8]. Now all these results can be extended to the quasimeromorphic case.

In our proof we shall use the methods of [10], [1], and [6, 2.3]. Notation and terminology will be as in [4], [5], and [9] but most of it will be explained in the course of the proof.

[†] Supported in part by the Samuel Neaman Fund, Special Year in Complex Analysis, Technion, I.I.T., Haifa, Israel, 1975/76.
Received October 19, 1976

2. Proof of the theorem

2.1. We may assume $f(0)=0$. For $r>0$ we let $U(r)$ denote the 0-component of $f^{-1}B^n(r)$. Note that since f is a local homeomorphism, f maps $\bar{U}(r)$ homeomorphically onto $\bar{B}^n(r)$ whenever $\bar{U}(r)\subset B^n$, see [6, lemma 2.2]. Let $r_0=\sup\{r>0:\bar{U}(r)\subset B^n\}$. Clearly $r_0>0$ and by Liouville's theorem for quasiregular mappings, see [4, 7.2], also $r_0<\infty$. Fix $r\in(0,r_0)$ and set $l^*=\inf\{|x|:x\in\partial U(r)\}$, $L^*=\sup\{|x|:x\in\partial U(r)\}$, and $l=\inf\{|f(x)|:|x|=l^*\}$. Since f is injective in $B^n(l^*)$, it suffices to find a lower bound for l^* in terms of n and $K=K(f)$.

2.2. Suppose that $l<r$. Then $A=U(r)\setminus\bar{U}(l)$ is a ring domain and f maps A quasiconformally onto the spherical ring $B^n(r)\setminus\bar{B}^n(l)$. Both boundary components of A meet the sphere $S^{n-1}(l^*)$, hence a well known capacity estimate for ring domains yields

$$\operatorname{cap}(U(r),\ \bar{U}(l))\geqq a_n>0$$

where a_n depends only on n, see [4, §5], [9, 11.7]. Thus by the capacity inequality for quasiconformal mappings

$$a_n\leqq\operatorname{cap}(U(r),\bar{U}(l))\leqq K\operatorname{cap}(B^n(r),\bar{B}^n(l))=K\omega_{n-1}(\log r/l)^{1-n}$$

where ω_{n-1} denotes the $(n-1)$-measure of S^{n-1}. This gives

$$r/l\leqq\alpha(n,K) \tag{2.3}$$

which is also true in the case $r=l$.

2.4. Pick $x_0\in\partial U(r)$ such that $|x_0|=L^*$ and set $y_0=f(x_0)$. Then $|y_0|=r$. For $t\in(0,r+l)$ and $\phi\in(0,\pi]$ we consider the spherical cap

$$C(t,\phi)=\{y:|y-y_0|=t,\qquad (y_0-y)\cdot y_0>rt\cos\phi\}$$

which lies on the sphere $S^{n-1}(y_0,t)$ and is symmetric with respect to the line segment $J=\{sy_0:-l/r\leqq s\leqq 1\}$ and meets J at the point $z_t=(r-t)y_0/r$. Let z_t^* be the unique point in $U(r)\cap f^{-1}(z_t)$ and let $C^*(t,\phi)$ be the z_t^*-component of $f^{-1}C(t,\phi)$. Set

$$\phi_t=\sup\{\phi\in(0,\pi]:f\text{ maps }C^*(t,\phi)\text{ homeomorphically onto }C(t,\phi)\}$$

and

$$I=\{t\in(0,r+l):C^*(t,\phi_t)\text{ meets }S^{n-1}(L^*)\}.$$

Note that $I \neq \emptyset$ since a neighborhood of x_0 is mapped by f homeomorphically onto a neighborhood of y_0. Hence $t_0 = \sup\{t : t \in I\} > 0$. It was shown in [6, p.5] that if $fB^n \subset R^n$ then $t_0 = r + l$ and $I \supset (r, r + l)$. In our case this cannot be done, however, we shall prove that $I \supset (0, t_0)$ and that if $t_0 < t < r + l$ then $\bar{C}^*(t, \phi_t)$ is the boundary of a domain D_t which is contained in $B^n(L^*)$ and mapped by f homeomorphically onto $\complement \bar{B}^n(y_0, t) = \bar{R}^n \setminus \bar{B}^n(y_0, t)$. To this end suppose that $t \in (0, r + l) \setminus I$. Then $C^*(t, \phi_t) \subset B^n(L^*)$. Now $C(t\phi_t)$ is simply connected, locally pathwise connected, and since $n \geqq 3$ also relatively locally connected, thus by [6, lemma 2.2] f maps $\bar{C}^*(t, \phi_t)$ homeomorphically onto $\bar{C}(t, \phi_t)$. At this point the proof breaks down for $n = 2$ because the punctured circle $C(t, \pi)$ is not relatively locally connected. Since $\bar{C}^*(t, \phi_t)$ is compact in B^n and f is a local homeomorphism and injective in $\bar{C}^*(t, \phi_t)$, it follows, see [10, remark 1, p. 422] that f is injective in a neighborhood of $\bar{C}^*(t, \phi_t)$. Consequently $\phi_t = \pi$ and this means that $\bar{C}^*(t, \phi_t)$ is a topological sphere. Thus the bounded component D_t of $\complement \bar{C}^*(t, \phi_t)$ is contained in $B^n(L^*)$. Now f maps D_t either onto $B^n(y_0, t)$ or onto $\complement \bar{B}^n(y_0, t)$, and since both domains are simply connected and f is a local homeomorphism sending ∂D_t injectively onto $S^{n-1}(y_0, t)$, the restriction $f \mid \bar{D}_t$ is a homeomorphism. But $fD_t = B^n(y_0, t)$ is impossible because f would be injective in $D_t \cup U(r)$ as $D_t \cap U(r) \neq \emptyset$ and $B^n(y_0, t) \cap B^n(r)$ is connected, see [10, remark 2, p. 422], while $f(x_0) = y_0$ and $x_0 \notin D_t$. Hence $fD_t = \complement \bar{B}^n(y_0, t)$. Furthermore, if $t' > t$ then $S^{n-1}(y_0, t') \subset fD_t$ and thus $\phi_{t'} = \pi$ and $\bar{C}^*(t', \phi_{t'}) \subset D_t$. Consequently, $t' \notin I$.

2.5 We have proved that

(i) I is an interval,

(ii) $(0, t_0) \subset I$, and

(iii) for $t_0 < t < r + l$, f maps D_t homeomorphically onto $\complement \bar{B}^n(y_0, t)$. Since $D_t \subset B^n(L^*)$ for $t_0 < t < r + l$, it easily follows that $\phi_{t_0} = \pi$ and the bounded component D_{t_0} of $\complement \bar{C}^*(t_0, \phi_{t_0})$ is contained in $B^n(L^*)$, ∂D_{t_0} meets $S^{n-1}(L^*)$, and f maps $\bar{D}_{t_0}$ homeomorphically onto $\complement B^n(y_0, t_0)$.

We consider two cases:

(a) $$t_0 \geqq r + l/2,$$

(b) $$t_0 < r + l/2.$$

2.6. *Case* (a). In this case the arguments will be similar to those of [6, 2.3]. Now by (ii) $C^*(t, \phi_t)$ meets $S^{n-1}(L^*)$ whenever $0 < t < r + l/2$. Set $V = \cup C(t, \phi_t)$ and $V^* = \cup C^*(t, \phi_t)$ where the unions are taken over all $t \in (r, r + l/2)$. Then V and V^* are domains and f maps V^* homeomorphically onto

V, cf. [10, p. 425]. For each $t \in (r, r + l/2)$ choose a point $x_t^* \in C^*(t, \phi_t) \cap S^{n-1}(L^*)$ and let $\Gamma(t)$ be the family of all paths joining x_t^* and z_t^* in $C^*(t, \phi_t)$. Finally set $\Gamma = \cup \Gamma(t)$. Since $|z_t^*| \leqq l^*$, we have

$$M(\Gamma) \leqq \omega_{n-1}(\log L^*/l^*)^{1-n}.$$

On the other hand, see [9, 10.12],

$$M(f\Gamma) \geqq b_n \log\left(\frac{r + l/2}{r}\right)$$

where $b_n > 0$ depends only on n. Now $M(f\Gamma) \leqq KM(\Gamma)$ yields

$$b_n \log(1 + l/(2r)) \leqq K\omega_{n-1}(\log L^*/l^*)^{1-n}.$$

Using (2.3) we get

$$b_n \log(1 + 1/(2\alpha(n, K)) \leqq K\omega_{n-1}(\log L^*/l^*)^{1-n}$$

which implies

$$l^* \geqq L^* \psi(n, K)$$

where $\psi(n, K) > 0$ depends only on n and K. Since $L^* \to 1$ as $r \to r_0$, the assertion for Case (a) follows.

2.7. *Case* (b). Choose a point $x_1 \in S^{n-1}(L^*) \cap \partial D_{t_0}$ and let $y_1 = f(x_1)$. Then the topological ball $U = D_{t_0} \cup U(r)$ is mapped by f homeomorphically onto $G = B^n(r) \cup \complement \bar{B}^n(y_0, t_0)$. Next we shall replace f by $g \circ f$ where $g : \bar{R}^n \to \bar{R}^n$ is quasiconformal and $gG = B^n(r)$. For this purpose we prove.

2.8. Lemma. *There exists a quasiconformal mapping $g : \bar{R}^n \to \bar{R}^n$ with maximal dilatation $K(g) \leqq 7^{n-1}$ such that $gG = B^n(r), g(0) = 0, g(y_0) = y_0$, and $g(y_1) = -y_0$.*

Proof of the Lemma. We may assume that $r = 1$ and $y_0 = e_1$ and that $y_1 = (a, b, 0, \cdots, 0)$ with $b \geqq 0$. Points in the 2-plane $P = \{x \in R^n : x_k = 0, 3 \leqq k \leqq n\}$ are treated as complex numbers $z = x_1 + ix_2$. The circles $C_0 = S^{n-1} \cap P$ and $C_1 = S^{n-1}(y_0, t_0) \cap P$ have two points of intersection z_0 and $\bar{z}_0$ with $\operatorname{Im} z_0 > 0$. Let S denote the sphere which is centered at the point λe_1 on the x_1-axis and which passes through the points z_0, $\bar{z}_0$, and 0; if z_0, $\bar{z}_0$ and 0 are colinear, then S denotes the $(n-1)$-plane $x_1 = 0$. Set $C_2 = S \cap P$ and let $\alpha < \pi/2$ denote the angle at which C_1 and C_0 meet at z_0. A look at the triangle $(0, y_0, z_0)$ shows that $t_0 < 3/2$ implies $\cos\alpha < 3/4$ and hence $2\pi/9 < \alpha < \pi/2$. Let $\beta < \pi$ denote the angle at which C_0 and C_2 meet at z_0. Then by considering the triangles $(\lambda e_1, 0, z_0)$ and $(0, y_0, z_0)$ it is not hard to see that $\beta = 2\alpha$. Thus $4\pi/9 < \beta < \pi$. Finally, let $\gamma = 2\pi - (\alpha + \beta)$, then $\pi/2 < \gamma < 4\pi/3$.

Let $A : \bar{R}^n \to \bar{R}^n$ be a Möbius-transformation which maps z_0 to 0, $\bar{z}_0$ to ∞, y_0 into itself, and the plane P in a sense-preserving way onto itself. Then $A \mid P$ is a linear fractional transformation, $|A(y_1)| \leqq 1$, $\arg A(y_1) = \alpha$, $|A(0)| = 1$, and $\arg A(0) = -\beta$. Furthermore, A maps G onto a domain bounded by two planes which meet along the $(n-2)$-plane $\{x \in R^n : x_1 = x_2 = 0\}$ at an angle $2\pi - \alpha$; the last angle is measured within AG.

Next we use cylindrical coordinates (r, ϕ, z) for points x in R^n; $r = \sqrt{x_1^2 + x_2^2}$, $\phi = \arg(x_1 + ix_2)$, and $z = (x_3, \cdots, x_n)$, and define a homeomorphism $\Phi : \bar{R}^n \to \bar{R}^n$ by

$$\begin{aligned} \Phi(r, \phi, z) &= (r, \pi\phi/2\beta, z) && \text{for } -\beta \leqq \phi < 0, \\ &= (r, \pi\phi/\alpha, z) && \text{for } 0 \leqq \phi < \alpha, \\ &= (r, \pi(1 - \alpha/2\gamma + \phi/2\gamma, z) && \text{for } \alpha \leqq \phi < \alpha + \gamma \end{aligned}$$

and $\Phi(\infty) = \infty$. Then Φ maps AG onto the half space $H = \{x \in R^n : x_2 < 0\}$, $\Phi(A(y_0)) = e_1$, $\Phi(A(0)) = -e_2$, and $\Phi(A(y_1)) = -\lambda e_1$ for some $\lambda \in [0, 1]$. Since $\pi/2\beta, \pi/\alpha, \pi/2\gamma \in [3/8, 9/2]$ it follows by [9, 16.3] that Φ is of maximal dilatation $K(\Phi) \leqq (9/2)^{n-1}$. Let $B : \bar{R}^n \to \bar{R}^n$ be a Möbius transformation such that $BH = H$, $BP = P$, $B(-e_2) = -e_2$ and $0 < B(y_0) = -B(-\lambda e_1)$ where $B(y_0)$ is treated again as a point in the complex plane P. The facts that $B \mid P$ is a linear fractional transformation and that $0 \leqq \lambda \leqq 1$ give $\sqrt{2} - 1 \leqq B(y_0) \leqq 1$.

The linear mapping $\Psi : \bar{R}^n \to \bar{R}^n$ defined by $\Psi(x) = (x_1/B(y_0), x_2, \cdots, x_n)$, $\Psi(\infty) = \infty$, is quasiconformal with maximal dilatation $K(\Psi) \leqq (1 + \sqrt{2})^{n-1}$. Finally let $C : \bar{R}^n \to \bar{R}^n$ be a Möbius transformation such that $CH = B^n$, $C(-e_2) = 0$, $C(e_1) = e_1$, and $C(-e_1) = -e_1$. Then $g = C \cdot \Phi \cdot B \cdot \Phi \cdot A$ has the required properties.

2.9. PROOF OF THE THEOREM—CONCLUSION. Let $F = g \cdot f$ with g as in Lemma 2.6. Then F maps $\bar{U}(r) \cup \bar{D}_{t_0} = \bar{U}$ homeomorphically onto $\bar{B}^n(r)$ and so U is the 0-component of $F^{-1}B^n(r)$. Furthermore, $F(x_0) = y_0$, $F(x_1) = -y_0$, $F(0) = 0$, and $K(F) \leqq K(g)K(f) \leqq 7^{n-1}K(f)$. To conclude the proof we shall find a lower bound for

$$l_1^* = \inf\{|x| : x \in \partial U\}.$$

The method will be the same as in Case (a).

Let

$$l_1 = \inf\{|F(x)| : |x| = l_1^*\},$$

then exactly as in 2.2 above

$$r/l_1 \leqq \alpha(n, 7^{n-1}K).$$

For $t \in (r, r + l_1)$ and $\phi \in (0, \pi)$ we consider the spherical cap $C(t, \phi)$ and its center z_t, as in 2.4. Let $z'_t = \bar{U} \cap F^{-1}(z_t)$, $C'(t, \phi)$ be the z'_t-component of $F^{-1}C(t, \phi)$ and $\phi_t = \sup\{\phi \in (0, \pi] : F$ maps $C'(t, \phi)$ homeomorphically onto $C(t, \phi)\}$. Now $C'(t, \phi_t)$ meets $S^{n-1}(L^*)$ for all $t \in (r, r + l_1)$ since otherwise, as in Case (a), $\phi_t = \pi$ and $\bar{C}'(t_1\phi_t)$ would be a topological sphere contained in $\bar{B}^n(L^*)$, which has to meet the set $E' = F^{-1}E$,

$$E = \{sy_0 : -1 \leqq s \leqq 1\},$$

at least twice while $\bar{C}(t, \phi_t)$ meets E at a single point. We can now proceed exactly as in Case (a) using path families on the caps $C'(t, \phi_t)$ and $C(t, \phi_t)$, respectively, for $t \in (r, r + l_1)$ and conclude

$$b_n \log(1 + 1/\alpha(n, 7^{n-1}K)) \leqq 7^{n-1}K\omega_{n-1} (\log L^*/l^*_1)^{1-n}.$$

This yields $l_1^* \geqq L^*\beta(n, K)$ where $\beta(n, K) > 0$ depends only on n and K. Letting $r \to r_0$ we have $L^* \to 1$ and thus F and so f is injective in $B^n(\beta(n, K))$. This proves Case (b) and the theorem follows.

2.10. REMARK. Let $\phi(n, K)$ and $\delta(n, K)$ denote the universal radius of injectivity for quasiregular, respectively, quasimeromorphic local homeomorphisms in B^n, $n > 2$. Clearly $\phi(n, K) \geqq \delta(n, K)$ with equality for $K = 1$. We do not know whether $\phi(n, K) > \delta(n, K)$ for any $K > 1$ and $n > 2$.

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